

EULER-MAHONIAN STATISTICS ON ORDERED SET PARTITIONS (II)

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ABSTRACT. We study statistics on ordered set partitions whose generating functions are related to p, q -Stirling numbers of the second kind. The main purpose of this paper is to provide bijective proofs of all the conjectures of Steingrímsson (Arxiv:math.CO/0605670). Our basic idea is to encode ordered partitions by a kind of path diagrams and explore the rich combinatorial properties of the latter structure. We also give a partition version of MacMahon's theorem on the equidistribution of the statistics inversion number and major index on words.

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1. INTRODUCTION

The systematic study of statistics on permutations and words has its origins in the work of MacMahon [15]. In this paper, we will consider MacMahon's three statistics for a word w : the number of *descents* ($\text{des } w$), the number of *inversions* ($\text{inv } w$), and the *major index* ($\text{maj } w$). These are defined as follows: A descent in a word $\pi = a_1 a_2 \cdots a_n$ is an i such that $a_i > a_{i+1}$, an inversion is a pair (i, j) such that $i < j$ and $a_i > a_j$, and the

major index of w is the sum of the descents in π . The rearrangement class $R(w)$ of a word $w = a_1 a_2 \cdots a_n$ is the set of all words obtained by permutating the letters of w .

Let $\mathbf{n} = (n_1, \dots, n_k)$ be a sequence of non negative integers and $R(\mathbf{n})$ the rearrangement class of the word $1^{n_1} \dots k^{n_k}$. Then MacMahon [1, Chap. 3] proved that

$$\sum_{w \in R(\mathbf{n})} q^{\text{inv } w} = \sum_{w \in R(\mathbf{n})} q^{\text{maj } w} = \frac{(q; q)_{n_1 + \dots + n_k}}{(q; q)_{n_1} \cdots (q; q)_{n_k}}, \quad (1.1)$$

where $(x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1})$. In particular, for the symmetric group S_n of $[n] := \{1, 2, \dots, n\}$, we have

$$\sum_{\sigma \in S_n} q^{\text{inv } \sigma} = \sum_{\sigma \in S_n} q^{\text{maj } \sigma} = [n]!_q, \quad (1.2)$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q [2]_q \cdots [n]_q$.

Any statistic that is equidistributed with des is said to be *Eulerian*, while any statistic equidistributed with inv is said to be *Mahonian*. A bivariate statistic (eul, mah) is said to be a *Euler-Mahonian* statistic if eul is Eulerian and mah is Mahonian.

An ordered partition of $[n]$ is a sequence of disjoint and nonempty subsets, called *blocks*, whose union is $[n]$. The blocks of an ordered partition will be written as capital letters separated by slashes, while elements of the blocks will be set in lower case. Thus an ordered partition of $[n]$ into k blocks is written as $\pi = B_1/B_2/\cdots/B_k$. Clearly we can identify an *unordered* partition with an ordered partition by arranging the blocks in the increasing order of their minima, called its *standard form*. For example, the partition π of $[9]$ consisting of the five blocks $\{1, 4, 7\}$, $\{2\}$, $\{3, 9\}$, $\{5\}$ and $\{6, 8\}$ will be written as $\pi = 1\ 4\ 7/2/3\ 9/5/6\ 8$. The set of all partitions of $[n]$ into k blocks will be denoted by \mathcal{P}_n^k . It is well-known that the *Stirling number of the second kind* $S(n, k)$ equals the cardinality of \mathcal{P}_n^k . Therefore, if \mathcal{OP}_n^k denotes the set of all ordered partitions of $[n]$ into k blocks, then $|\mathcal{OP}_n^k| = k! S(n, k)$.

For any ordered partition $\pi \in \mathcal{OP}_n^k$ there is a unique partition $\pi_0 = B_1/B_2/\cdots/B_k \in \mathcal{P}_n^k$ and a unique permutation $\sigma \in S_k$ such that $\pi = B_{\sigma(1)}/B_{\sigma(2)}/\cdots/B_{\sigma(k)}$. In parallel with notion of σ -restricted growth function in [21], we shall call the corresponding partition π a σ -partition. Let $\mathcal{P}_n^k(\sigma)$ be the set of all σ -partitions of $[n]$ into k blocks. For instance, $\pi = 6\ 8/5/1\ 4\ 7/3\ 9/2 \in \mathcal{P}_9^5(\sigma)$ with $\sigma = 54132$.

Clearly, for any $\sigma \in S_k$ we have $|\mathcal{P}_n^k(\sigma)| = |\mathcal{P}_n^k| = S(n, k)$ and $\mathcal{P}_n^k = \mathcal{P}_n^k(\varepsilon)$ where ε is the *identity permutation*.

The p, q -Stirling numbers of the second kind $S_{p,q}(n, k)$ were introduced in [22] by the recursion:

$$S_{p,q}(n, k) = \begin{cases} p^{k-1} S_{p,q}(n-1, k-1) + [k]_{p,q} S_{p,q}(n-1, k), & \text{if } 0 < k \leq n; \\ 1, & \text{if } n = k = 0; \\ 0, & \text{otherwise.} \end{cases} \quad (1.3)$$

where

$$[k]_{p,q} = p^{k-1} + p^{k-2}q + \cdots + pq^{k-1} + q^{k-1}.$$

When p or q is set to 1, we obtain two usual q -Stirling numbers of the second kind (see [10]):

$$S_q(n, k) := S_{q,1}(n, k) \quad \text{and} \quad \widetilde{S}_q(n, k) := S_{1,q}(n, k) = q^{-\binom{k}{2}} S_q(n, k) \quad (1.4)$$

Many authors (see e.g. [6, 9, 11, 13, 16, 17, 18, 19, 21, 22, 23]) have explored the combinatorial aspects of these q -Stirling numbers. In particular, Wachs and White [22] studied the combinatorial interpretations of these p, q -Stirling numbers. In a sequel paper Wachs [21] extended some unordered partition interpretations of p, q -Stirling number of the second kind to σ -partition statistics, although she used the *restricted growth functions* instead of set partitions.

A statistic STAT on ordered set partitions is *Euler-Mahonian* if for any $n \geq k \geq 1$ its distribution over \mathcal{OP}_n^k equals $[k]_q! S_q(n, k)$, i.e.,

$$\sum_{\pi \in \mathcal{OP}_n^k} q^{\text{STAT } \pi} = [k]_q! S_q(n, k).$$

In [19] Steingrímsson conjectured several hard Euler-Mahonian statistics on \mathcal{OP}_n^k . In a previous paper [11], Ishikawa and the two current authors proved half of the conjectures of Steingrímsson [19] by using the *Matrix-transfer method* and determinant computations.

The aim of this paper is to give a complete bijective approach to Steingrímsson's problem. In particular, we will not only derive the results in [11] bijectively but also settle the remaining half of the conjectures. In fact our bijective approach yields also new results on p, q -Stirling numbers of Wachs and White [22] and the σ -partitions of Wachs [21]. As we will show, one of our results generalizes MacMahon's equidistribution result of inversion number and major index on words.

Throughout this paper, we shall denote by \mathbb{P} (resp. \mathbb{N} , \mathbb{Z}) the set of positive integers (resp. non negative integers, integers) and assume that n and k are two fixed integers satisfying $n \geq k \geq 1$. Furthermore, for any integers i_1, i_2, \dots, i_k , we denote by $\{i_1, i_2, \dots, i_k\}_<$ (resp. $\{i_1, i_2, \dots, i_k\}_>$) the increasing (resp. decreasing) arrangement of these integers.

2. DEFINITIONS

Let B be a finite subset of \mathbb{N} . The *opener* of B is its least element while the *closer* of B is its greatest element. For $\pi \in \mathcal{OP}_n^k$, we will denote by $\text{open}(\pi)$ and $\text{clos}(\pi)$ the sets of openers and closers of the blocks of π , respectively. The letters (integers) in π are further divided into four classes:

- *singletons*: elements of the singleton blocks;
- *strict openers*: smallest elements of the non singleton blocks;
- *strict closers*: largest elements of the non singleton blocks;
- *transients*: all other elements, i.e., non extremal elements of non singleton blocks.

The sets of strict openers, strict closers, singletons and transients of π will be denoted, respectively, by $\emptyset(\pi)$, $\mathcal{C}(\pi)$, $\mathcal{S}(\pi)$ and $\mathcal{T}(\pi)$. Obviously we have

$$\text{open}(\pi) = \emptyset(\pi) \cup \mathcal{S}(\pi), \quad \text{clos}(\pi) = \mathcal{C}(\pi) \cup \mathcal{S}(\pi), \quad \mathcal{S}(\pi) = \text{open}(\pi) \cap \text{clos}(\pi).$$

The 4-tuple $(\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi), \mathcal{T}(\pi))$, denoted by $\lambda(\pi)$, is called the *type* of π . For example, if $\pi = 3\ 5/2\ 4\ 6/1/7\ 8$, then $\text{open}(\pi) = \{1, 2, 3, 7\}$, $\text{clos}(\pi) = \{1, 5, 6, 8\}$ and

$$\lambda(\pi) = (\{2, 3, 7\}, \{5, 6, 8\}, \{1\}, \{4\}).$$

Let $\mathcal{OP}_n^k(\lambda)$ be the set of ordered partitions in \mathcal{OP}_n^k of type λ .

Following Steingrímsson [19], we now define a system of ten inversion-like statistics on ordered set partitions. Most of them have been studied in the case of set partitions. Note that the two last statistics were essentially defined by Foata and Zeilberger [7] for permutations. The reader is referred to [13, 19, 21] for further informations of these statistics.

Given a partition $\pi = B_1/B_2/\dots/B_k \in \mathcal{OP}_n^k$, let w_i be the index of the block (counting from left to right) containing i , namely the integer j such that $i \in B_j$. Then define ten coordinate statistics as follows. For $1 \leq i \leq n$, we let:

$$\begin{aligned} \text{los}_i \pi &= \#\{j \in \text{open } \pi \mid j < i, w_j < w_i\}, \\ \text{ros}_i \pi &= \#\{j \in \text{open } \pi \mid j < i, w_j > w_i\}, \\ \text{lob}_i \pi &= \#\{j \in \text{open } \pi \mid j > i, w_j < w_i\}, \\ \text{rob}_i \pi &= \#\{j \in \text{open } \pi \mid j > i, w_j > w_i\}, \\ \text{lcs}_i \pi &= \#\{j \in \text{clos } \pi \mid j < i, w_j < w_i\}, \\ \text{rcs}_i \pi &= \#\{j \in \text{clos } \pi \mid j < i, w_j > w_i\}, \\ \text{lcb}_i \pi &= \#\{j \in \text{clos } \pi \mid j > i, w_j < w_i\}, \\ \text{rcb}_i \pi &= \#\{j \in \text{clos } \pi \mid j > i, w_j > w_i\}. \end{aligned}$$

Moreover, let $\text{rsb}_i \pi$ (resp. $\text{lsb}_i \pi$) be the number of blocks B in π to the right (resp. left) of the block containing i such that the opener of B is smaller than i and the closer of B is greater than i . Remark that lsb_i and rsb_i are each equal to the difference of two of the first eight statistics. Namely, it is easy to see that

$$\text{lsb}_i = \text{los}_i - \text{lcs}_i = \text{lcb}_i - \text{lob}_i \quad \text{and} \quad \text{rsb}_i = \text{ros}_i - \text{rcs}_i = \text{rcb}_i - \text{rob}_i. \quad (2.1)$$

Then define the statistics ros , rob , rcs , rcb , lob , los , lcs , lcb , lsb and rsb as the sum of their coordinate statistics, e.g.

$$\text{ros} = \text{ros}_1 + \dots + \text{ros}_n.$$

For any partition π we can define the restrictions of these statistics on openers and non openers $\text{ros}_{\mathcal{OS}}$, $\text{ros}_{\mathcal{TC}}$, ..., $\text{rsb}_{\mathcal{OS}}$ and $\text{rsb}_{\mathcal{TC}}$, e.g.

$$\text{ros}_{\mathcal{OS}} \pi = \sum_{i \in \mathcal{O} \cup \mathcal{S}(\pi)} \text{ros}_i \pi \quad \text{and} \quad \text{ros}_{\mathcal{TC}} \pi = \sum_{i \in \mathcal{T} \cup \mathcal{C}(\pi)} \text{ros}_i \pi. \quad (2.2)$$

Remark 2.1. Note that ros is the abbreviation of "right, opener, smaller", while lcb is the abbreviation of "left, closer, bigger", etc.

As an example, we give here the values of the coordinate statistics computed on $\pi = 6\ 8/5/1\ 4\ 7/3\ 9/2$:

$$\begin{array}{rcl}
\pi = & 6\ 8 & / \ 5 \ / \ 1\ 4\ 7 \ / \ 3\ 9 \ / \ 2 \\
\\
\text{los}_i : & 0\ 0 & / \ 0 \ / \ 0\ 0\ 2 \ / \ 1\ 3 \ / \ 1 \\
\text{ros}_i : & 4\ 4 & / \ 3 \ / \ 0\ 2\ 2 \ / \ 1\ 1 \ / \ 0 \\
\text{lob}_i : & 0\ 0 & / \ 1 \ / \ 2\ 2\ 0 \ / \ 2\ 0 \ / \ 3 \\
\text{rob}_i : & 0\ 0 & / \ 0 \ / \ 2\ 0\ 0 \ / \ 0\ 0 \ / \ 0 \\
\text{lcs}_i : & 0\ 0 & / \ 0 \ / \ 0\ 0\ 1 \ / \ 0\ 3 \ / \ 0 \\
\text{rcs}_i : & 2\ 3 & / \ 1 \ / \ 0\ 1\ 1 \ / \ 1\ 1 \ / \ 0 \\
\text{lcb}_i : & 0\ 0 & / \ 1 \ / \ 2\ 2\ 1 \ / \ 3\ 0 \ / \ 4 \\
\text{rcb}_i : & 2\ 1 & / \ 2 \ / \ 2\ 1\ 1 \ / \ 0\ 0 \ / \ 0 \\
\text{lsb}_i : & 0\ 0 & / \ 0 \ / \ 0\ 0\ 1 \ / \ 1\ 0 \ / \ 1 \\
\text{rsb}_i : & 2\ 1 & / \ 2 \ / \ 0\ 1\ 1 \ / \ 0\ 0 \ / \ 0
\end{array}$$

It follows that $\text{ros}_{\mathcal{OS}} \pi = 8$ and $\text{rsb}_{\mathcal{TC}} \pi = 3$.

The following result is due to Wachs and White [22, Cor. 5.3].

$$\sum_{\pi \in \mathcal{P}_n^k} p^{\text{rcb } \pi} q^{\text{lsb } \pi} = S_{p,q}(n, k). \quad (2.3)$$

Let $\pi = B_1/B_2/\dots/B_k \in \mathcal{OP}_n^k$. Define a *partial order* \succ on blocks B_i 's as follows : $B_i \succ B_j$ if all the letters of B_i are greater than those of B_j , i.e., if $\min(B_i) > \max(B_j)$. A *block inversion* in π is a pair (i, j) such that $i < j$ and $B_i \succ B_j$. We denote by $\text{bInv } \pi$ the number of block inversions in π . A *block descent* is an integer i such that $B_i \succ B_{i+1}$. The *block major index* of π , denoted by $\text{bMaj } \pi$, is the sum of the block descents in π . We also define their complementary counterparts:

$$\text{cbInv} = \binom{k}{2} - \text{bInv} \quad \text{cbMaj} = \binom{k}{2} - \text{bMaj}, \quad (2.4)$$

and two composed statistics:

$$\begin{aligned}
\text{cinvLSB} &= \text{lsb} + \text{cbInv} + \binom{k}{2}, \\
\text{cmajLSB} &= \text{lsb} + \text{cbMaj} + \binom{k}{2}.
\end{aligned} \quad (2.5)$$

For any σ -partition $\pi \in \mathcal{P}_n^k(\sigma)$ with $\sigma \in \mathcal{S}_k$, we can define two natural statistics :

$$\text{Inv } \pi = \text{inv } \sigma \quad \text{and} \quad \text{Maj } \pi = \text{maj } \sigma. \quad (2.6)$$

The following two statistics are the (ordered) partition analogues of their counterparts in permutations [19, p.13]:

$$\text{MAK} = \text{ros} + \text{lcs}, \quad \text{MAK}' = \text{lob} + \text{rcb}. \quad (2.7)$$

In what follows we will also denote by MAG any of these two statistics, i.e.,

$$\text{MAG} \in \{\text{MAK}, \text{MAK}'\}. \quad (2.8)$$

3. MAIN RESULTS

We first present our main result on the equidistribution of some inversion like statistics on σ -partitions.

Theorem 3.1. *For any $\sigma \in \mathcal{S}_k$, the triple statistics $(\text{MAK} + \text{bInv}, \text{MAK}' + \text{bInv}, \text{cinvLSB})$ and $(\text{MAK}' + \text{bInv}, \text{MAK} + \text{bInv}, \text{cinvLSB})$ are equidistributed on $\mathcal{P}_n^k(\sigma)$. Moreover,*

$$\sum_{\pi \in \mathcal{P}_n^k(\sigma)} p^{(\text{MAG} + \text{bInv})\pi} q^{\text{cinvLSB } \pi} = q^{k(k-1)} \left(\frac{p}{q}\right)^{\text{inv } \sigma} S_{p,q}(n, k). \quad (3.1)$$

Note that (3.1) gives two σ -extensions of (2.3), which is the $\sigma = \epsilon$ case of (3.1).

As inv is a Mahonian statistic on S_k , summing the two sides over all the permutations σ in S_k , we derive immediately from (3.1) the main result of [11], of which the second part was conjectured by Steingrímsson [19].

Theorem 3.2 (Ishikawa et al.). *We have*

$$\sum_{\pi \in \mathcal{OP}_n^k} p^{(\text{MAG} + \text{bInv})\pi} q^{\text{cinvLSB } \pi} = q^{\binom{k}{2}} [k]_{p,q}! S_{p,q}(n, k). \quad (3.2)$$

In particular the three statistics

$$\text{MAK} + \text{bInv}, \quad \text{MAK}' + \text{bInv}, \quad \text{cinvLSB} \quad (3.3)$$

are Euler-Mahonian on \mathcal{OP}_n^k .

For the major like statistics we have the following equidistribution result on ordered partitions with a fixed type.

Theorem 3.3. *The triple statistics $(\text{MAK} + \text{bMaj}, \text{MAK}' + \text{bMaj}, \text{cmajLSB})$ and $(\text{MAK} + \text{bInv}, \text{MAK}' + \text{bInv}, \text{cinvLSB})$ are equidistributed on $\mathcal{OP}_n^k(\lambda)$ for any partition type λ .*

Combining Theorems 3.2 and 3.3 we derive immediately the following result, of which the first part was conjectured by Ishikawa et al. [11, Conjecture 6.2] while the second part was originally conjectured by Steingrímsson [19].

Theorem 3.4. *We have*

$$\sum_{\pi \in \mathcal{OP}_n^k} p^{(\text{MAG} + \text{bMaj})\pi} q^{\text{cmajLSB } \pi} = q^{\binom{k}{2}} [k]_{p,q}! S_{p,q}(n, k). \quad (3.4)$$

In particular the three statistics

$$\text{MAK} + \text{bMaj}, \quad \text{MAK}' + \text{bMaj}, \quad \text{cmajLSB} \quad (3.5)$$

are Euler-Mahonian on \mathcal{OP}_n^k .

For any set partition $\pi \in \mathcal{P}_n^k$, denote by $\mathcal{R}(\pi)$ the rearrangement class of π , i.e., if $\pi = B_1/B_2/\cdots/B_k$, then

$$\mathcal{R}(\pi) = \{B_{\sigma(1)}/B_{\sigma(2)}/\cdots/B_{\sigma(k)} \mid \sigma \in \mathcal{S}_k\}.$$

For instance, if $\pi = 1\ 4/2\ 3/5$, then

$$\mathcal{R}(\pi) = \{1\ 4/2\ 3/5, 1\ 4/5/2\ 3, 5/2\ 3/1\ 4, 2\ 3/1\ 4/5, 2\ 3/5/1\ 4, 5/1\ 4/2\ 3\}.$$

It is clear that $|\mathcal{R}(\pi)| = k!$ for any partition π with k blocks.

Introduce first two analogues of inversion numbers and major index on \mathcal{OP}_n^k :

$$\begin{aligned} \text{INV} &= \text{rsb}_{\mathcal{OS}} + \text{bInv}, \\ \text{MAJ} &= \text{rsb}_{\mathcal{OS}} + \text{bMaj}. \end{aligned} \quad (3.6)$$

The statistic INV is just a rewording of the inversion number inv in (2.6). Indeed, it is easy to see that $\text{bInv} = \text{rcs}_{\mathcal{OS}}$ and $\text{Inv} = \text{ros}_{\mathcal{OS}}$, therefore

$$\text{INV} = \text{rsb}_{\mathcal{OS}} + \text{rcs}_{\mathcal{OS}} = \text{ros}_{\mathcal{OS}} = \text{Inv}. \quad (3.7)$$

Our last result is a non trivial extension of MacMahon's identity (1.1) to rearrangement class of an arbitrary partition.

Theorem 3.5. *For any $\pi \in \mathcal{P}_n^k$, the statistics INV and MAJ are equidistributed on $\mathcal{R}(\pi)$ and*

$$\sum_{\pi \in \mathcal{R}(\pi)} q^{\text{MAJ} \pi} = \sum_{\pi \in \mathcal{R}(\pi)} q^{\text{INV} \pi} = [k]_q!. \quad (3.8)$$

To show that (3.8) implies MacMahon's formula (1.1), we consider the rearrangement class of a special set partition as follows. Let $N_i = n_1 + \dots + n_i$ for $i = 1, \dots, k$ and

$$\Pi = \pi_{11} \dots \pi_{1n_1} \pi_{21} \dots \pi_{2n_2} \dots \pi_{k1} \dots \pi_{kn_k}$$

be the partition of $[2N_k]$ consisting of the doubletons:

$$\pi_{ij} := \{2N_{i-1} + j, 2N_{i-1} + n_i + j\}, \quad (1 \leq i \leq k, 1 \leq j \leq n_i, N_0 = 0).$$

It is readily seen that each $\pi \in \mathcal{R}(\Pi)$ can be identified with a pair $(w, (\pi_1, \dots, \pi_k))$ where $w \in R(\mathbf{n})$ is the word obtained from π by substituting each π_{ij} by i for $1 \leq i \leq k$ and π_i is the word obtained from π by deleting all the π_{lj} for $l \neq i$. For example, if $k = 3$, $n_1 = 3$, $n_2 = 2$ and $n_3 = 3$ then

$$\Pi = \{1, 4\} \{2, 5\} \{3, 6\} \{7, 9\} \{8, 10\} \{11, 14\} \{12, 15\} \{13, 16\}.$$

Let $\pi \in \mathcal{R}(\Pi)$ be the ordered partition:

$$\pi = \{7, 9\} \{2, 5\} \{11, 14\} \{1, 4\} \{13, 16\} \{12, 15\} \{3, 6\} \{8, 10\}.$$

Then $\pi \mapsto (w, (\pi_1, \pi_2, \pi_3))$ with $w = 21313321$ and

$$\pi_1 = \{2, 5\} \{1, 4\} \{3, 6\}, \pi_2 = \{7, 9\} \{8, 10\}, \pi_3 = \{11, 14\} \{13, 16\} \{12, 15\}.$$

Note that for any i, j, l, r such that $1 \leq i, j \leq k$, $1 \leq l \leq n_i$ and $1 \leq r \leq n_j$,

$$\pi_{il} \prec \pi_{jr} \iff i < j,$$

therefore

$$\text{bMaj} \pi = \text{maj} w, \quad \text{bInv} \pi = \text{inv} w$$

and

$$\text{rsb}_{\mathcal{OS}} \pi = \text{rsb}_{\mathcal{OS}} \pi_1 + \dots + \text{rsb}_{\mathcal{OS}} \pi_k.$$

It then follows from (3.6) that

$$\sum_{\pi \in \mathcal{R}(\Pi)} q^{\text{MAJ}\pi} = \left(\sum_{w \in \mathcal{R}(\mathbf{n})} q^{\text{maj } w} \right) \prod_{i=1}^k \sum_{\pi \in \mathcal{R}(\pi_i)} q^{\text{rsb}_{\mathcal{OS}} \pi}, \quad (3.9)$$

$$\sum_{\pi \in \mathcal{R}(\Pi)} q^{\text{INV}\pi} = \left(\sum_{w \in \mathcal{R}(\mathbf{n})} q^{\text{inv } w} \right) \prod_{i=1}^k \sum_{\pi \in \mathcal{R}(\pi_i)} q^{\text{rsb}_{\mathcal{OS}} \pi}. \quad (3.10)$$

As $\text{rcs}_{\mathcal{OS}}(\pi) = 0$ for any $\pi \in \mathcal{R}(\pi_i)$ we have $\text{INV} = \text{rsb}_{\mathcal{OS}}$ on $\mathcal{R}(\pi_i)$. Hence, by Theorem 3.5,

$$\sum_{\pi \in \mathcal{R}(\pi_i)} q^{\text{rsb}_{\mathcal{OS}} \pi} = [n_i]_q!, \quad 1 \leq i \leq k,$$

and

$$\sum_{\pi \in \mathcal{R}(\Pi)} q^{\text{INV}\pi} = [N_k]_q!.$$

MacMahon's formula (1.1) follows then from (3.9) and (3.10) by invoking Theorem 3.5.

The rest of this paper is organized as follows. In Section 4 we shall present the first path diagrams encoding of ordered partitions $\Phi : \Delta_n^k \longrightarrow \mathcal{OP}_n^k$, which was introduced in [11]. We will construct an involution on path diagrams $\varphi : \Delta_n^k \longrightarrow \Delta_n^k$ in Section 5 and a bijection Γ_σ from set partitions to ordered set partitions in Section 6. We prove Theorem 3.1 in section 7. To deal with major like statistics a second path diagram encoding $\Psi : \Delta_n^k \longrightarrow \mathcal{OP}_n^k$ will be given in Section 8. Then, in Section 9 we prove Theorem 3.4 by using the mapping $\Upsilon := \Psi \circ \Phi^{-1} : \mathcal{OP}_n^k \rightarrow \mathcal{OP}_n^k$. Finally we prove Theorem 3.6 in Section 3.6 and conclude the paper with some further remarks.

4. THE FIRST PATH DIAGRAM ENCODING Φ OF ORDERED PARTITIONS

As shown in [11], we can define the notion of *trace* or *skeleton* for ordered partitions. To this end, adjoin to \mathbb{P} a symbol ∞ such that $i < \infty$ for any positive integer i . The restriction of a subset B of \mathbb{P} on $[i]$, namely $B \cap [i]$, is said to be

- *empty* if $i < \min B$,
- *active* if $\min B \leq i < \max B$,
- *complete* if $\max B \leq i$.

Let $\pi \in \mathcal{OP}_n^k$. For $1 \leq i \leq n$ the i -trace or i -skeleton T_i of π is obtained by restricting each block on $[i]$ and deleting empty blocks. By convention, we add a symbol ∞ at the end of each active block. By convention, the latter is still called block.

Clearly one can characterize a partition and the statistics ros and rsb by using its traces. More precisely, for any $1 \leq i \leq n$, $\text{ros}_i \pi$ (resp. $\text{rsb}_i \pi$) equals the number of blocks (resp. active blocks) to the right of the block containing i in the i -trace T_i of π .

Example 4.1. If $\pi = 3 \ 5 \ 7/1 \ 4 \ 10/9/6/2 \ 8$, then the 7-trace of π is

$$T_7 = 3 \ 5 \ 7/1 \ 4 \ \infty/6/2 \ \infty,$$

with $\text{ros}_7 \pi = 3$ and $\text{rsb}_7 \pi = 2$.

A useful way to describe the traces of a partition is to draw a *path diagram*.

Definition 4.1. A path of depth k and length n is a sequence $w = (w_0, w_1, \dots, w_n)$ of points in \mathbb{N}^2 such that $w_0 = (0, 0)$, $w_n = (k, 0)$ and the i -th ($1 \leq i \leq n$) step (w_{i-1}, w_i) must be one of the following four types:

- North, i.e., $w_i - w_{i-1} = (0, 1)$,
- East, i.e., $w_i - w_{i-1} = (1, 0)$,
- South-East, i.e., $w_i - w_{i-1} = (1, -1)$,
- Null, i.e., $w_i - w_{i-1} = (0, 0)$ and $y_i > 0$ if $w_i = (x_i, y_i)$.

The abscissa and ordinate of w_{i-1} are called the abscissa and height of the i -th step of w and denoted by $x_i(w)$ and $y_i(w)$, respectively. The set of all paths of depth k and length n will be denoted by Ω_n^k .

We can visualize a path w by drawing a *segment* or *loop* from w_{i-1} to w_i in the xy plane. For instance, the path

$$w = ((0, 0), (0, 1), (0, 2), (0, 3), (0, 3), (0, 3), (1, 3), (2, 2), (3, 1), (4, 1), (5, 0)),$$

is illustrated in Figure 1.

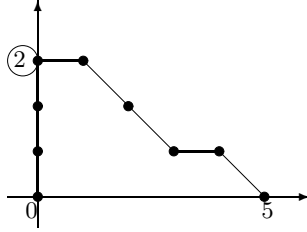


FIGURE 1. A path in Ω_{10}^5 with two successive Null steps from $(0, 3)$ to $(0, 3)$.

For the reason which will be clear later, the sets of indices of North, South-East, East and Null steps of a path w will be denoted by $\mathcal{O}(w)$, $\mathcal{C}(w)$, $\mathcal{S}(w)$ and $\mathcal{T}(w)$, respectively. The 4-tuple $(\mathcal{O}(w), \mathcal{C}(w), \mathcal{S}(w), \mathcal{T}(w))$, denoted by $\lambda(w)$, is called the *type* of w . For instance, if w is the path represented in Figure 1, then

$$\lambda(w) = (\{1, 2, 3\}, \{7, 8, 10\}, \{6, 9\}, \{4, 5\}).$$

Definition 4.2. A path diagram of depth k and length n is a pair (w, γ) , where w is a path in Ω_n^k and $\gamma = (\gamma_i)_{1 \leq i \leq n}$ is a sequence of integers such that

- $0 \leq \gamma_i \leq y_i(w) - 1$ if the i -th step of w is Null or South-East,
- $0 \leq \gamma_i \leq x_i(w) + y_i(w)$ if the i -th step of w is North or East.

Let Δ_n^k be the set of path diagrams of depth k and length n .

For any statement A we denote by $\chi(A)$ the character function of A , that means $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false.

Lemma 4.3. Let $w \in \Omega_n^k$ and $(w, \gamma) \in \Delta_n^k$ with $\mathcal{O} \cup \mathcal{S}(w) = \{i_1, i_2, \dots, i_k\}_{<}$. Then

- $x_i + y_i = x_{i-1} + y_{i-1} + \chi(i \in \mathcal{O} \cup \mathcal{S}(w))$ for $i = 1, \dots, n$.
- $x_{i_j} + y_{i_j} + 1 = j$ for $j = 1, 2, \dots, k$.

Proof. Since $x_1(w) + y_1(w) = 0$ and each step is one of the four kinds: $(0, 1)$, $(1, 0)$, $(0, 0)$ and $(1, -1)$, the sum $x_i + y_i$ increases by one if and only if the i -th step is North or South. This yields (i), while (ii) is a direct consequence of (i). \square

One can encode ordered partitions by path diagrams. The following important bijection $\Phi : \Delta_n^k \rightarrow \mathcal{OP}_n^k$ was introduced in [11].

Algorithm Φ : Starting from a path diagram $h = (w, \gamma)$ in Δ_n^k , we obtain $\Phi(h) = \pi$ by constructing recursively all the i -traces T_i ($1 \leq i \leq n$) of π , i.e., such that $\pi = T_n$. By convention $T_0 = \emptyset$. Assume that we have constructed $T_{i-1} = B_{i_1}/B_{i_2}/\dots/B_{i_\ell}$ such that T_{i-1} has $y_i(w)$ active blocks and $x_i(w)$ complete blocks. Label the slots before B_{i_1} , between B_{i_j} and $B_{i_{j+1}}$, for $1 \leq j \leq \ell - 1$, and after B_{i_ℓ} from left to right by $\ell, \dots, 1, 0$, while the active blocks of T_{i-1} are labeled from left to right by $y_i(w) - 1, \dots, 1, 0$. Extend T_{i-1} to T_i as follows:

- If the i -th step of w is North (resp. East), then create an active block (resp. singleton) with i at the slot with label γ_i ;
- If the i -th step of w is Null (resp. South-East), then insert i (resp. replace ∞ by i) in the active block with label γ_i .

Since $x_{n+1}(w) = k$ and $y_{n+1}(w) = 0$, the n -trace T_n is a partition in \mathcal{OP}_n^k .

Example 4.2. Consider the path diagram $h = (w, \gamma) \in \Delta_{10}^5$, where w is the path in Figure 1 and $\gamma = (0, 0, 2, 1, 2, 3, 2, 0, 1, 0)$, then $\Phi(h) = 6/3 \ 5 \ 7/1 \ 4 \ 10/9/2 \ 8$. The step by step construction of $\Phi(h)$ is given in Figure 2, where the i -th step is labeled with γ_i for $1 \leq i \leq 10$.

The main properties of Φ are given in the following theorem of Ishikawa et al [11].

Theorem 4.4. The bijection $\Phi : (w, \gamma) \mapsto \pi$ has the following properties: $\lambda(w) = \lambda(\pi)$ and for $1 \leq i \leq n$,

$$\gamma_i = \begin{cases} \text{ros}_i(\pi), & \text{if } i \in \mathcal{O}(\pi) \cup \mathcal{S}(\pi); \\ \text{rsb}_i(\pi), & \text{if } i \in \mathcal{T}(\pi) \cup \mathcal{C}(\pi). \end{cases}$$

It follows that

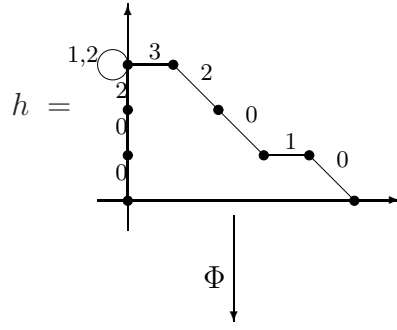
$$\text{ros}_{\mathcal{OS}} \pi = \sum_{i \in \mathcal{O} \cup \mathcal{S}(w)} \gamma_i \quad \text{and} \quad \text{rsb}_{\mathcal{TC}} \pi = \sum_{i \in \mathcal{T} \cup \mathcal{C}(w)} \gamma_i. \quad (4.1)$$

5. INVOLUTION φ ON PATH DIAGRAMS

For any path $w \in \Omega_n^k$, one can define a natural bijection between the North steps and South-East steps of w . More precisely, let

$$\mathcal{O}(w) = \{o_1, o_2, \dots, o_r\}_< \quad \text{and} \quad \mathcal{C}(w) = \{c_1, c_2, \dots, c_r\}_<.$$

We define the permutation $\sigma \in S_r$, called the *associated permutation* of w , as follows: Suppose the height of the i -th North step of w (i.e. the o_i -th step of w) is t . Since $w_0 = (0, 0)$ and $w_n = (k, 0)$, there must exist a South-East step of height $t + 1$ to the right of the o_i -th step. If the first such step is the j -th South-East step (i.e. the c_j -th step of w), set $\sigma(i) = j$. Since there is at least one South-East step of height $t + 1$ between any



i	step_i	γ_i	T_i
0			\emptyset
1	<i>North</i>	0	1 ∞
2	<i>North</i>	0	$1 \infty / \mathbf{2} \infty$
3	<i>North</i>	2	3 $\infty / 1 \infty / 2 \infty$
4	<i>Null</i>	1	$3 \infty / 1 \mathbf{4} \infty / 2 \infty$
5	<i>Null</i>	2	3 5 , $\infty / 1 \mathbf{4}, \infty / 2 \infty$
6	<i>East</i>	3	6/3 5 $\infty / 1 \mathbf{4} \infty / 2 \infty$
7	<i>South-East</i>	2	$6/3 \mathbf{5 7} / 1 \mathbf{4} \infty / 2 \infty$
8	<i>South-East</i>	0	$6/3 \mathbf{5 7} / 1 \mathbf{4} \infty / 2 \mathbf{8}$
9	<i>East</i>	1	$6/3 \mathbf{5 7} / 1 \mathbf{4} \infty / \mathbf{9} / 2 \mathbf{8}$
10	<i>South-East</i>	0	$6/3 \mathbf{5 7} / 1 \mathbf{4} \mathbf{10} / 9 / 2 \mathbf{8}$.

FIGURE 2. the step by step construction of $\Phi(h)$

two North steps of height t the mapping σ is an injection and then a bijection for $\mathcal{O}(w)$ and $\mathcal{C}(w)$ have the same cardinality.

For instance, if w is the path represented below, we have $\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(3) = 1$ and $\sigma(4) = 3$. The construction of σ is illustrated in Figure 3.

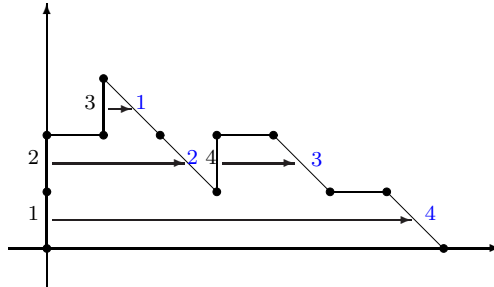


FIGURE 3. Construction of an associated permutation

For a subset $A \subseteq [n]$, the complement set \bar{A} is obtained by replacing each $i \in A$ by $\bar{i} := n + 1 - i$. The *reverse path* of w is the path \bar{w} whose i -th step is North (resp. East, Null, South-East) if and only if the $(n + 1 - i)$ -th step of w is South-East (resp. East, Null, North). Clearly, if $w \in \Omega_n^k$ with $\lambda(w) = (\emptyset, \mathcal{C}, \mathcal{S}, \mathcal{T})$, then we can also define \bar{w} as the unique path satisfying $\lambda(\bar{w}) = \overline{\lambda(w)} := (\overline{\mathcal{C}}, \overline{\emptyset}, \overline{\mathcal{S}}, \overline{\mathcal{T}})$.

Lemma 5.1. *The mapping $w \mapsto \bar{w}$ is an involution on Ω_n^k . Moreover,*

(1) *For $i \in [n]$, we have $y_i(\bar{w}) = y_{\bar{i}+1}(w)$. In particular, we have:*

$$y_i(\bar{w}) = \begin{cases} y_{\bar{i}}(w) - 1, & \text{if } i \in \emptyset(\bar{w}); \\ y_{\bar{i}}(w), & \text{if } i \in \mathcal{T} \cup \mathcal{S}(\bar{w}); \\ y_{\bar{i}}(w) + 1, & \text{if } i \in \mathcal{C}(\bar{w}). \end{cases} \quad (5.1)$$

(2) *Suppose $|\emptyset(w)| = r$ and let σ and σ' be the associated permutations of w and \bar{w} respectively. Then for any $j \in [r]$,*

$$\sigma'(j) = r + 1 - \sigma^{-1}(r + 1 - j). \quad (5.2)$$

Proof. (1) By definition of \bar{w} , the height of the i -step of \bar{w} corresponds to the height of the $\bar{i} + 1$ -th step of w , so $y_i(\bar{w}) = y_{\bar{i}+1}(w)$. Eq. (5.1) follows then from the fact that $\lambda(\bar{w}) = \overline{\lambda(w)}$ and the equation:

$$y_{i+1}(w) = \begin{cases} y_i(w) + 1 & \text{if } i \in \emptyset(w); \\ y_i(w) & \text{if } i \in \mathcal{T} \cup \mathcal{S}(w); \\ y_i(w) - 1 & \text{if } i \in \mathcal{C}(w). \end{cases}$$

(2) By definition, for any $j \in [r]$, the mapping σ maps the j -th North step of w to the $\sigma(j)$ -th South-East step of w . Equivalently the mapping σ' maps the $r + 1 - \sigma(j)$ -th North step of \bar{w} to the $r + 1 - j$ -th South-East step of \bar{w} . In other words, we have $\sigma'(r + 1 - \sigma(j)) = r + 1 - j$. Substituting $r + 1 - \sigma(j)$ by i yields the desired result. \square

We have now all the ingredients to define our involution φ on Δ_n^k .

Involution φ . Let $h = (w, \gamma) \in \Delta_n^k$ such that $\emptyset \cup \mathcal{S}(w) = \{i_1, i_2, \dots, i_k\}_<$, $\mathcal{T}(w) = \{t_1, t_2, \dots, t_u\}_<$ and $\mathcal{C}(w) = \{c_1, c_2, \dots, c_r\}_<$. If $\emptyset \cup \mathcal{S}(\bar{w}) = \{i'_1, i'_2, \dots, i'_k\}_<$, $\mathcal{T}(\bar{w}) = \{t'_1, t'_2, \dots, t'_u\}_<$ and $\mathcal{C}(\bar{w}) = \{c'_1, c'_2, \dots, c'_r\}_<$, then $\varphi(h) = (\bar{w}, \xi)$, where $\xi = (\xi_i)_{1 \leq i \leq n}$ is defined as follows:

$$\xi_i = \begin{cases} \gamma_{i_m}, & \text{if } i = i'_m \text{ for } m \in [k]; \\ \gamma_{t_{u+1-m}}, & \text{if } i = t'_m \text{ for } m \in [u]; \\ \gamma_{c_{\sigma(r+1-m)}}, & \text{if } i = c'_m \text{ for } m \in [r]; \end{cases} \quad (5.3)$$

where σ is the associated permutation of w .

Example 5.1. Consider the path diagram $h = (w, \gamma)$ in the Figure 4. It is easy to see that the permutation associated to w is $\sigma = 321$. It follows that $\xi_{c'_1} = \gamma_{c_{\sigma(3)}} = \gamma_{c_1}$, $\xi_{c'_2} = \gamma_{c_{\sigma(2)}} = \gamma_{c_2}$ and $\xi_{c'_3} = \gamma_{c_{\sigma(1)}} = \gamma_{c_3}$. The image $\varphi(h)$ of h is given below.

We now present the main result of this section.

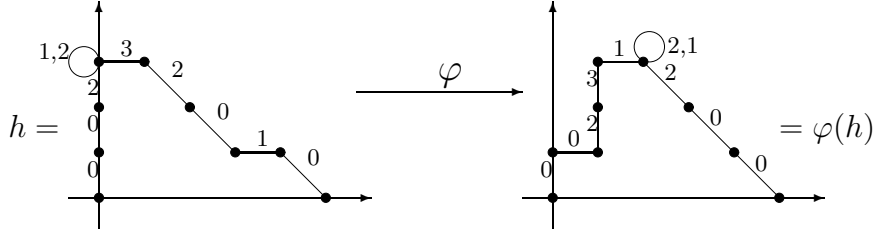


FIGURE 4. An example of the involution φ

Proposition 5.2. *The mapping $\varphi : (w, \gamma) \mapsto (\bar{w}, \xi)$ is an involution on Δ_n^k such that*

$$\sum_{i \in \emptyset \cup S(\bar{w})} \xi_i = \sum_{i \in \emptyset \cup S(w)} \gamma_i \quad \text{and} \quad \sum_{i \in T \cup C(\bar{w})} \xi_i = \sum_{i \in T \cup C(w)} \gamma_i. \quad (5.4)$$

Proof. We first show that the mapping φ is well defined. It suffices to show that:

- (a) $0 \leq \xi_{i'_m} \leq x_{i'_m}(\bar{w}) + y_{i'_m}(\bar{w})$ for $m \in [k]$,
- (b) $0 \leq \xi_{t'_m} \leq y_{t'_m}(\bar{w}) - 1$ for $m \in [u]$, and
- (c) $0 \leq \xi_{c'_m} \leq y_{c'_m}(\bar{w}) - 1$ for $m \in [r]$.

Since $t'_m = \overline{t_{u+1-m}}$ and $c'_m = \overline{c_{r+1-m}}$, we have

$$\overline{t'_m} = \overline{\overline{t_{u+1-m}}} = t_{u+1-m} \quad \text{and} \quad \overline{c'_m} = \overline{\overline{c_{r+1-m}}} = c_{r+1-m}.$$

- (i) For $m \in [k]$, Lemma 4.3(ii) implies that $x_{i'_m}(\bar{w}) + y_{i'_m}(\bar{w}) = x_{i_m}(w) + y_{i_m}(w)$ because both sides are equal to $m - 1$. We have (a) by invoking $\xi_{i'_m} = \gamma_{i_m}$ and $0 \leq \gamma_{i_m} \leq x_{i_m}(w) + y_{i_m}(w)$.
- (ii) For $m \in [u]$, Lemma 5.1(2) implies that $y_{t'_m}(\bar{w}) = y_{\overline{t'_m}}(w) = y_{t_{u+1-m}}(w)$. We obtain (b) by invoking $\xi_{t'_m} = \gamma_{t_{u+1-m}}$, $0 \leq \gamma_{t_{u+1-m}} \leq y_{t_{u+1-m}}(w) - 1$ and $y_{t_{u+1-m}}(w) = y_{t'_m}(\bar{w})$.
- (iii) For $m \in [r]$, Lemma 5.1(2) implies $y_{c'_m}(\bar{w}) = y_{c_{\sigma(r+1-m)}}(w)$ because $y_{c'_m}(\bar{w}) = y_{\overline{c'_m}}(w) + 1 = y_{\overline{c_{r+1-m}}}(w) + 1 = y_{c_{\sigma(r+1-m)}}(w) + 1$, which is equal to $y_{c_{\sigma(r+1-m)}}(w)$ by definition of σ . We derive (c) by invoking that $\xi_{c'_m} = \gamma_{c_{\sigma(r+1-m)}}$ and $0 \leq \gamma_{c_{\sigma(r+1-m)}} \leq y_{c_{\sigma(r+1-m)}}(w) - 1$.

Let σ' be the associated permutation of \bar{w} . Consider the chain of bijections:

$$\varphi^2 : (w, \gamma) \xrightarrow{\varphi} (\bar{w}, \gamma) \xrightarrow{\varphi} (\overline{\bar{w}}, \mu) = (w, \mu)$$

where $\mu = (\mu_i)_{1 \leq i \leq n}$ is defined by

$$\mu_i = \begin{cases} \xi_{i'_m}, & \text{if } i = i_m \text{ for } m \in [k]; \\ \xi_{t'_{u+1-m}}, & \text{if } i = t_m \text{ for } m \in [u]; \\ \xi_{c'_{\sigma'(r+1-m)}}, & \text{if } i = c_m \text{ for } m \in [r]. \end{cases}$$

Therefore

$$\mu_{i_m} = \xi_{i'_m} = \gamma_{i_m} \text{ for } m \in [k] \quad \text{and} \quad \mu_{t_m} = \xi_{t'_{u+1-m}} = \gamma_{t_m} \text{ for } m \in [u].$$

By (5.2) we have $\sigma(r+1-\sigma'(j)) = r+1-j$ and

$$\mu_{c_m} = \xi_{c'_{\sigma'(r+1-m)}} = \gamma_{c_{\sigma(r+1-\sigma'(r+1-m))}} = \gamma_{c_m} \quad \text{for } m \in [r].$$

Hence $\mu = \gamma$ and φ is an involution. Finally (5.4) follows from (5.3). \square

6. BIJECTION Γ_σ FROM \mathcal{P}_n^k TO $\mathcal{P}_n^k(\sigma)$

The *Lehmer code* of a permutation σ in S_n is the sequence $c(\sigma) = (c_1, \dots, c_n)$ of non negative integers where the integer c_i is defined by

$$c_i = \#\{j > i, \sigma(j) < \sigma(i)\}.$$

We can recover the permutation σ from its code $c(\sigma)$ because $\sigma(i)$ equals the $(c_i + 1)$ -th element in $[n] - \{\sigma(1), \dots, \sigma(i-1)\}$. Therefore the mapping c which associates to each permutation of S_n its Lehmer code is a bijection from S_n to $[0, n-1] \times [0, n-2] \times \dots \times [0, 1] \times [0]$. Moreover $\text{inv } \sigma = c_1 + \dots + c_n$. For our purpose we need to define the d -code of a permutation $\sigma \in S_n$ by

$$d(\sigma) := (d_1, \dots, d_n) = (c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(n)}).$$

That is, the coordinate d_i is the number of entries $\sigma(j)$ smaller than and to the right of i in the sequence $\sigma(1) \dots \sigma(n)$. In other words, we have

$$d_i = \#\{j > \sigma^{-1}(i); \sigma(j) < i\}.$$

Lemma 6.1. *The mapping d which associates to each permutation of S_n its d -code is a bijection from S_n to $P_n = [0] \times [0, 1] \times \dots \times [0, n-2] \times [0, n-1]$. Moreover $\text{inv } \sigma = d_1 + \dots + d_n$.*

For example, if $\sigma = 86347521$, then

$$c(\sigma) = 75223210,$$

$$d(\sigma) = 01222537.$$

Lemma 6.2. *For any $(w, \gamma) \in \Delta_n^k$ with $\emptyset \cup \mathcal{S}(w) = \{i_1, i_2, \dots, i_k\}_<$, the sequence $(\gamma_{i_1}, \dots, \gamma_{i_k})$ is the d -code of some permutation $\sigma \in S_k$.*

Proof. It suffices to verify that $0 \leq \gamma_{i_j} \leq j-1$ for $j = 1, \dots, k$, but this is obvious in view of Lemma 4.3(ii). \square

For any permutation $\sigma \in S_k$, let $\Delta_n^k(\sigma)$ be the set of path diagrams $(w, \gamma) \in \Delta_n^k$ such that $d(\sigma) = (\gamma_{i_1}, \dots, \gamma_{i_k})$, where $\emptyset \cup \mathcal{S}(w) = \{i_1, i_2, \dots, i_k\}_<$.

Lemma 6.3. *For any $\sigma \in S_k$ we have*

- (i) *the restriction of φ on $\Delta_n^k(\sigma)$ is an involution;*
- (ii) *the restriction of Φ on $\Delta_n^k(\sigma)$ is a bijection from $\Delta_n^k(\sigma)$ to $\mathcal{P}_n^k(\sigma)$.*

Proof. Let $(w, \gamma) \in \Delta_n^k(\sigma)$ with $\emptyset \cup \mathcal{S}(w) = \{i_1, i_2, \dots, i_k\}_\leq$. Then $(\gamma_{i_1}, \dots, \gamma_{i_k})$ is the d -code of σ . (i) follows directly from the definition of φ . Let $\Phi(w, \gamma) = \pi$. Suppose $\pi = B_{\tau(1)}/B_{\tau(2)}/\dots/B_{\tau(k)}$ is a τ -partition in \mathcal{OP}_n^k and $d(\tau) = (d_1, \dots, d_k)$. Hence $B_1/B_2/\dots/B_k$ is a partition in \mathcal{P}_n^k and $\min B_j = i_j$ for $j \in [k]$. By definition of Φ , in the i_j -th step, we create a new block B_j with i_j in T_{i_j} so that there are γ_{i_j} blocks to the right of the block B_j , i.e., $d_j = \gamma_{i_j}$. Namely $\tau = \sigma$.

Conversely, given $\pi \in \mathcal{P}_n^k(\sigma)$, then there is a path diagram $(w, \gamma) \in \Delta_n^k(\tau)$, for some $\tau \in S_k$, such that $\Phi(w, \gamma) = \pi$. As Φ is a bijection, we get $d(\sigma) = d(\tau)$, so $\sigma = \tau$. This completes the proof of (ii). \square

It is convenient to introduce the following abbreviations:

$$\text{cls} := \text{lcs} + \text{rcs}, \quad \text{opb} := \text{lob} + \text{rob} \quad \text{and} \quad \text{sb} := \text{lsb} + \text{rsb}. \quad (6.1)$$

Proposition 6.4. *For any $\pi \in \mathcal{OP}_n^k$,*

$$\text{cls}(\pi) = \sum_{i \in \text{clos}(\pi)} (n - i), \quad (6.2)$$

$$\text{opb}(\pi) = \sum_{i \in \text{open}(\pi)} (i - 1), \quad (6.3)$$

$$\text{sb}(\pi) = \sum_{i \in \mathcal{C}(\pi)} i - \sum_{i \in \mathcal{O}(\pi)} i + k - n. \quad (6.4)$$

Proof. First of all, equations (6.2) and (6.3) follow immediately from the fact that for any $i \in [n]$ the number of integers greater (resp. smaller) than i in $[n]$ equals $n - i$ (resp. $i - 1$). Indeed, by definition (6.1), the statistic $\text{cls}(\pi)$ amounts to count, for each closer i of π , the number of the integers greater than i in $[n]$. Similarly we get (6.3). Now, suppose $\pi = B_1 / \cdots / B_k$, then

$$\begin{aligned} \text{sb}(\pi) &= \sum_{i=1}^k |\{j : \min(B_i) < j < \max(B_i), j \notin B_i\}| \\ &= \sum_{i=1}^k (\max(B_i) - \min(B_i) + 1 - |B_i|) \\ &= \sum_{i=1}^k \max(B_i) - \sum_{i=1}^k \min(B_i) + k - n, \end{aligned}$$

which is exactly (6.4). \square

We are now ready to construct a bijection from (no ordered) set partitions to ordered set partitions and state the main theorem of this section.

Theorem 6.5. *For any $\sigma \in S_k$ there is a bijection $\Gamma_\sigma : \mathcal{P}_n^k \rightarrow \mathcal{P}_n^k(\sigma)$ such that for any $\pi \in \mathcal{P}_n^k$,*

- (i) $\lambda(\Gamma_\sigma(\pi)) = \lambda(\pi)$;
- (ii) $(\text{cls}, \text{opb}, \text{sb})\Gamma_\sigma(\pi) = (\text{cls}, \text{opb}, \text{sb})\pi$;
- (iii) $\text{rsb}_{\mathcal{TC}} \Gamma_\sigma(\pi) = \text{rsb}_{\mathcal{TC}} \pi$.

Proof. Let σ, τ be two permutations in S_k with $d(\sigma) = (d_1, \dots, d_k)$ and $d(\tau) = (d'_1, \dots, d'_k)$. For any path diagram $(w, \gamma) \in \Delta_n^k(\tau)$ we can define a path diagram $g_{\tau, \sigma}(w, \gamma) = (w, \gamma') \in \Delta_n^k(\sigma)$ as follows:

$$\gamma'_i = \begin{cases} d_j, & \text{if } i = i_j \in \emptyset \cup \mathcal{S}(w); \\ \gamma_i, & \text{if } i \in \mathcal{T} \cup \mathcal{C}(w). \end{cases} \quad (6.5)$$

Clearly the mapping $g_{\sigma, \tau} : \Delta_n^k(\sigma) \rightarrow \Delta_n^k(\tau)$ is a bijection because $g_{\sigma, \tau}^{-1} = g_{\tau, \sigma}$.

In particular, taking $\tau = \epsilon$ (the identity permutation), then $g_\sigma := g_{\epsilon, \sigma}$ is a bijection from $\Delta_n^k(\epsilon)$ to $\Delta_n^k(\sigma)$. It follows that

$$\Gamma_\sigma := \Phi \circ g_\sigma \circ \Phi^{-1} \quad (6.6)$$

is a bijection from \mathcal{P}_n^k to $\mathcal{P}_n^k(\sigma)$. For any $\pi \in \mathcal{P}_n^k$, let $\Gamma_\sigma(\pi) = \pi'$. The composition of mappings is better understood by the following diagram:

$$\Gamma_\sigma : \pi \xrightarrow{\Phi^{-1}} (w, \gamma) \xrightarrow{g_\sigma} (w, \gamma') \xrightarrow{\Phi} \pi'.$$

By definition of Φ and g_σ , we have $\lambda(\pi') = \lambda(w) = \lambda(\pi)$, namely (i). (ii) follows from Proposition 6.4. Combining (4.1) and (6.5) yields (iii):

$$\text{rsb}_{\mathcal{TC}} \pi' = \sum_{i \in \mathcal{TUC}(w)} \gamma'_i = \sum_{i \in \mathcal{TUC}(w)} \gamma_i = \text{rsb}_{\mathcal{TC}} \pi.$$

This completes the proof of Theorem 6.5.

Example 6.1. Let $\pi = 1\ 5\ 7/2\ 4\ 10/3\ 8/6/9 \in \mathcal{P}_{10}^5$ and $\sigma = 4\ 3\ 1\ 5\ 2$. Hence $\text{opener}(\pi) = \{1, 2, 3, 6, 9\}_<$ and $d(\sigma) = 00231$. The corresponding path diagrams (w, γ) and (w, γ') are given by

$$\begin{aligned} w &= (0, 0), (0, 1), (0, 2), (0, 3), (0, 3), (0, 3), (1, 3), (2, 2), (3, 1), (4, 1), (5, 0), \\ \gamma &= (\mathbf{0}, \mathbf{0}, \mathbf{0}, 1, 2, \mathbf{0}, 2, 0, \mathbf{0}, 0), \\ \gamma' &= (\mathbf{0}, \mathbf{0}, \mathbf{2}, 1, 2, \mathbf{3}, 2, 0, \mathbf{1}, 0). \end{aligned}$$

The mapping $g_\sigma : h \mapsto h'$ is illustrated in Figure 5.

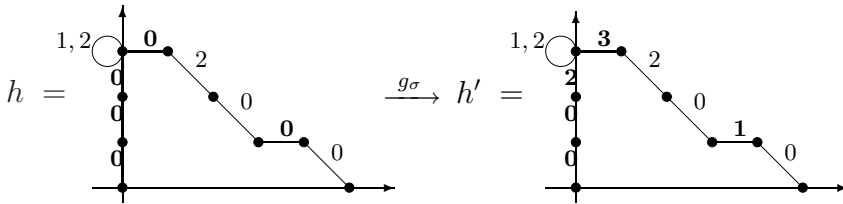


FIGURE 5. Mapping g_σ

Finally, we get $\Gamma_\sigma(\pi) = \Phi(h') = 6/3 \ 5 \ 7/1 \ 4 \ 10/9/2 \ 8 \in \mathcal{P}_{10}^5(\sigma)$. Note that π' is not a rearrangement of the blocks in π .

7. PROOF OF THEOREM 3.1

Consider the mapping

$$\Xi = \Phi \circ \varphi \circ \Phi^{-1} : \mathcal{OP}_n^k \longrightarrow \mathcal{OP}_n^k.$$

For example, if $\pi = 6/3 \ 5 \ 7/1 \ 4 \ 10/9/2 \ 8$, then $\Phi^{-1}(\pi) = h$ is given in Figure 2, while $\varphi(h)$ is given in Figure 4. Finally we get $\Xi(\pi) = \Phi(\varphi(h)) = 4 \ 6 \ 8/3 \ 7 \ 10/1 \ 9/5/2$.

Clearly the mapping Ξ is an involution. For any fixed $\sigma \in \mathcal{S}_k$, Lemma 6.3 implies that the restriction of Ξ on $\mathcal{P}_n^k(\sigma)$ is stable.

For any $\pi \in \mathcal{P}_n^k(\sigma)$, let $\Xi(\pi) = \pi' \in \mathcal{P}_n^k(\sigma)$. Suppose that $h = (w, \gamma) = \Phi^{-1}(\pi)$, $h' = (\bar{w}, \xi) = \varphi(h)$ and $\pi' = \Phi(h')$. Then Lemmas 4.3 and 6.3 imply that

$$\text{Inv } \pi = \text{inv } \sigma = \text{Inv } \pi'. \quad (7.1)$$

By Theorem 4.4 we see that $\lambda(\pi') = \lambda(\bar{w})$, $\lambda(\pi) = \lambda(w)$ and

$$\text{rsb}_{\mathcal{TC}} \pi' = \sum_{i \in T \cup \mathcal{C}(\bar{w})} \xi_i \quad \text{and} \quad \text{rsb}_{\mathcal{TC}} \pi = \sum_{i \in T \cup \mathcal{C}(w)} \gamma_i.$$

It follows from (5.4) that

$$\text{rsb}_{\mathcal{TC}} \pi' = \text{rsb}_{\mathcal{TC}} \pi. \quad (7.2)$$

As $\lambda(\bar{w}) = \overline{\lambda(w)}$, by Proposition 6.4, we obtain

$$(\text{cls}, \text{opb}, \text{sb})\pi' = (\text{opb}, \text{cls}, \text{sb})\pi. \quad (7.3)$$

Furthermore, on \mathcal{OP}_n^k , the following equations hold true (see [11, Lemma 4.6]):

$$\begin{aligned} \text{MAK} + \text{bInv} &= \text{cls} + \text{rsb}_{\mathcal{TC}} + \text{Inv}, \\ \text{MAK}' + \text{bInv} &= \text{opb} + \text{rsb}_{\mathcal{TC}} + \text{Inv}, \\ \text{cinvLSB} &= k(k-1) + \text{sb} - \text{rsb}_{\mathcal{TC}} - \text{Inv}. \end{aligned} \quad (7.4)$$

It follows from (7.2), (7.3) and (7.4) that

$$\begin{aligned} &(\text{MAK} + \text{bInv}, \text{MAK}' + \text{bInv}, \text{cinvLSB}) \pi' \\ &= (\text{MAK}' + \text{bInv}, \text{MAK} + \text{bInv}, \text{cinvLSB}) \pi. \end{aligned}$$

This completes the first part of Theorem 3.1.

In view of (7.4) it is easy to see that (3.1) is equivalent to the following two identities:

$$\sum_{\pi \in \mathcal{P}_n^k(\sigma)} p^{\text{cls } \pi + \text{rsb}_{\mathcal{TC}} \pi} q^{\text{sb } \pi - \text{rsb}_{\mathcal{TC}} \pi} = \sum_{\pi \in \mathcal{P}_n^k(\sigma)} p^{\text{opb } \pi + \text{rsb}_{\mathcal{TC}} \pi} q^{\text{sb } \pi - \text{rsb}_{\mathcal{TC}} \pi}, \quad (7.5)$$

and

$$\sum_{\pi \in \mathcal{P}_n^k(\sigma)} p^{\text{opb } \pi + \text{rsb}_{\mathcal{TC}} \pi} q^{\text{sb } \pi - \text{rsb}_{\mathcal{TC}} \pi} = S_{p,q}(n, k). \quad (7.6)$$

Now, (7.5) follows immediately from (7.2) and (7.3). According to Theorem 6.5, it suffices to prove (7.6) in the case of $\sigma = \varepsilon$. But, on $\mathcal{P}_n^k = \mathcal{P}_n^k(\varepsilon)$, since $\text{lob} = \text{rsb}_{\mathcal{OS}} = 0$, there hold

$$\begin{aligned} \text{opb} + \text{rsb}_{\mathcal{TC}} &= \text{rob} + \text{rsb} = \text{rcb}, \\ \text{sb} - \text{rsb}_{\mathcal{TC}} &= \text{lsb} = \text{lsb} + \text{lob} = \text{lcb}. \end{aligned}$$

Hence the left-hand side of (7.6) is equal to

$$\sum_{\pi \in \mathcal{P}_n^k} p^{\text{opb } \pi + \text{rsb}_{\mathcal{TC}} \pi} q^{\text{sb } \pi - \text{rsb}_{\mathcal{TC}} \pi} = \sum_{\pi \in \mathcal{P}_n^k} p^{\text{rcb } \pi} q^{\text{lcb } \pi}, \quad (7.7)$$

and (7.6) follows by applying (2.3). The proof of Theorem 3.1 is thus completed.

Remark 7.1. For any mahonian permutation statistic mah we can define a statistic Mah on ordered set partitions by $\text{Mah } \pi = \text{mah } \sigma$ if $\pi \in \mathcal{P}_n^k(\sigma)$ and derive from the above proof the following identities:

$$\begin{aligned} \sum_{\pi \in \mathcal{OP}_n^k} p^{\text{cls } \pi + \text{rsb}_{\mathcal{TC}} \pi} q^{\text{sb } \pi - \text{rsb}_{\mathcal{TC}} \pi} t^{\text{Mah } \pi} &= [k]_t! S_{p,q}(n, k), \\ \sum_{\pi \in \mathcal{OP}_n^k} p^{\text{opb } \pi + \text{rsb}_{\mathcal{TC}} \pi} q^{\text{sb } \pi - \text{rsb}_{\mathcal{TC}} \pi} t^{\text{Mah } \pi} &= [k]_t! S_{p,q}(n, k). \end{aligned} \quad (7.8)$$

In particular, by taking $\text{mah} = \text{inv}$ we recover essentially Theorem 3.2, while by taking $\text{mah} = \text{maj}$ we derive that the statistics

$$\text{cls} + \text{rsb}_{\mathcal{TC}} + \text{Maj}, \quad \text{opb} + \text{rsb}_{\mathcal{TC}} + \text{Maj}, \quad k(k-1) + \text{sb} - \text{rsb}_{\mathcal{TC}} - \text{Maj}$$

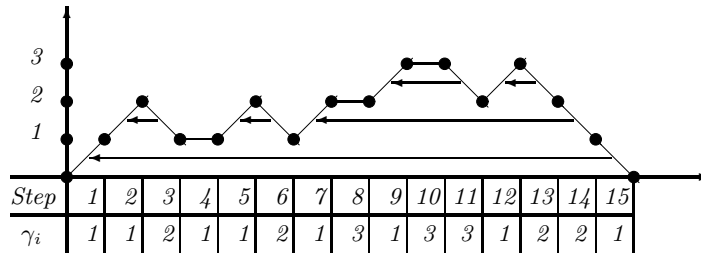
are Euler-Mahonian.

Remark 7.2. For ordinary partitions, there is a similar bijection, simpler than Ξ , using Motzkin paths. We sketch this bijection below.

A Motzkin path of length n is a lattice path in the plane of integer lattice \mathbb{Z}^2 from $(0, 0)$ to $(n, 0)$, consisting of NE-steps $(1, 1)$, E-steps $(1, 0)$, and SE-steps $(1, -1)$, which never passes below the x -axis. Let \mathcal{D}_n be the set of Motzkin path diagrams (ω, γ) , where ω is a Motzkin path of length n and $\gamma = (\gamma_1, \dots, \gamma_n)$ is a sequence of labels such that if the i -th step is NE, then $\gamma_i = 1$, if the i -th step is SE, then $1 \leq \gamma_i \leq h_i$, and if the i -th step is E, then $1 \leq \gamma_i \leq h_i + 1$, where h_i is the height of the i -th step.

For each partition π in \mathcal{P}_n we can construct its traces $\{T_i\}_{0 \leq i \leq n}$. Let $\gamma_i - 1$ be the number of incomplete blocks to the left of the block containing i in T_i . As shown in [13] we can construct a bijection $f : \pi \mapsto (\omega, \gamma)$ from \mathcal{P}_n to \mathcal{D}_n as follows: For $i = 1, \dots, n$, if $i \in \mathcal{O}(\pi)$ (resp. $\mathcal{S}(\pi)$) we draw a NE-step (resp. E-step) with label 1 (resp. γ_i) and if $i \in \mathcal{C}(\pi)$ (resp. $\mathcal{T}(\pi)$) we draw a SE-step (resp. E-step) with label γ_i .

An example is given in Figure 6.



$$\pi = \{1, 4, 15\} / \{2, 3\} / \{5, 6\} / \{7, 10, 13\} / \{8\} / \{9, 11\} / \{12, 14\}.$$

FIGURE 6. A labeled Motzkin path of length 15 and the corresponding partition.

Define an involution $g : (\omega, \gamma) \mapsto (\omega', \gamma')$ on \mathcal{D}_n as follows: First reverse the path w by reading it from right to left, i.e., if $w = ((i, y_i))_{0 \leq i \leq n}$, then $w' = ((i, y_{n-i}))_{0 \leq i \leq n}$, then pair the NE-steps with SE-steps in w two by two in the following way: each NE-step at height h corresponds to the first SE-step to its right at height $h+1$ (thus we establish a bijection between the SE-steps of w and those of w'), attribute the label of each SE-step of w to the

corresponding SE-step of w' , finally the labels of NE-steps of w' are 1 and the E-steps of w' keep the same label as in w .

Now, it is easy to see (cf. [13]) that the mapping

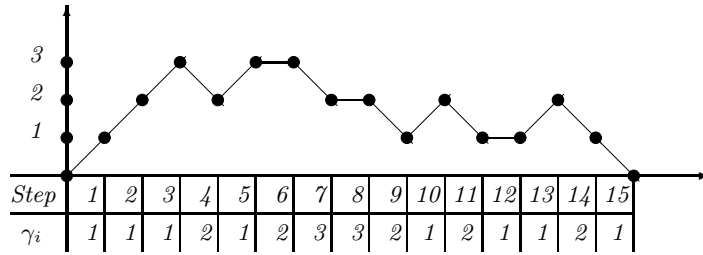
$$\Lambda = f^{-1} \circ g \circ f : \pi \xrightarrow{f} (w, \gamma) \xrightarrow{g} (w', \gamma') \xrightarrow{f^{-1}} \pi'$$

is an involution on \mathcal{P}_n^k such that

$$\text{MAK}(\pi) = \text{rcb } \Lambda(\pi) \quad \text{and} \quad \text{lcb}(\Lambda(\pi)) = \text{lcb}(\pi).$$

The involution applied to the example in Figure 1 is given in Figure 2, where

$$\text{MAK}\pi = \text{rcb } \Lambda(\pi) = 37, \quad \text{lcb } \pi = \text{lcb } \Lambda(\pi) = 16.$$



$$\Lambda(\pi) = \{1, 12, 15\} / \{2, 4\} / \{3, 6, 9\} / \{5, 7\} / \{8\} / \{10, 11\} / \{13, 14\}.$$

FIGURE 7. The labeled Motzkin path and the associated partition corresponding to those of Figure 6.

Note that the mapping $\Lambda = f^{-1} \circ g \circ f$ is a corrected version of that given in [13].

8. THE SECOND PATH DIAGRAMS ENCODING Ψ OF ORDERED SET PARTITIONS

Recall that a block in a trace is a subset B of $[n] \cup \{\infty\}$ such that $B \cap [n] \neq \emptyset$. By convention, the closer of B is the greatest element of B . Hence $\max(B) = \infty$ if $\infty \in B$. Thus, the statistics bDes , bMaj and rsb_i can be easily extended to traces.

Let $T = B_1 / \cdots / B_r$ be the $(i-1)$ -trace of a partition ($i \geq 1$). We can insert i before B_1 , between two adjacent blocks B_j and B_{j+1} , for $1 \leq j \leq r-1$, or after B_r . Label these insertion positions from left to right by $0, 1, \dots, r$. We say that the position j is *active* if B_{j+1} is active.

Let A and D be the set of active positions and block descents in T , respectively. We then label the $r+1$ positions in T as follows:

- label the right-most position by $a_0 = r$ and the elements of $A \cup D$ from right to left by a_1, a_2, \dots, a_t . So $a_t < a_{t-1} < \cdots < a_2 < a_1$,
- label the remaining positions from left to right by a_{t+1}, \dots, a_r . So $a_{t+1} < \cdots < a_r$.

Lemma 8.1. *Let ℓ be an integer satisfying $0 \leq \ell \leq r$ and define T' to be the i -trace obtained by inserting in T the block $\{i, \infty\}$ or $\{i\}$ into position a_ℓ . Then,*

$$\text{rsb}_i T' + \text{bMaj } T' - \text{bMaj } T = \ell. \quad (8.1)$$

Proof. Clearly A and D are disjoint. Suppose $A = \{o_1, o_2, \dots, o_m\}_>$ and $D = \{d_1, d_2, \dots, d_p\}_>$. So $t = m + p$. Let B be one of the two blocks $\{i, \infty\}$ or $\{i\}$.

Since T is a $(i - 1)$ -trace, the openers of T are all smaller than i . This implies that $\text{rsb}_i T'$ is the number of active blocks in T' to the right of B . We distinguish four cases.

- (1) $\ell = 0$. Since $a_0 = r$, we have $T' = B_1/B_2/\dots/B_r/B$ and the equation (8.1) is obvious.
- (2) $1 \leq \ell \leq t$ and $a_\ell \in D$. Let $a_\ell = d_j$ for some j , $1 \leq j \leq p$. The block descent set of T' is then

$$\{d_p < \dots < d_{j+1} < d_j + 1 < d_{j-1} + 1 < \dots < d_1 + 1\}.$$

Let q be the greatest number such that $o_q > a_\ell$. Clearly, $\text{rsb}_i(T') = q$. We thus have that

$$\text{rsb}_i T' + \text{bMaj } T' - \text{bMaj } T = q + j.$$

It suffices now to remark that $\ell = j + q$.

- (3) $1 \leq \ell \leq t$ and $a_\ell \in A$. Then, $a_\ell = o_j$ for some j , $1 \leq j \leq m$. Clearly, $\text{rsb}_i(T') = j$. Let q be the greatest number such that $d_q > a_\ell$. Then, the block descent set of T' is

$$\{d_p < \dots < d_{q+1} < d_q + 1 < d_{q-1} + 1 < \dots < d_1 + 1\}.$$

It then follows that

$$\text{rsb}_i T' + \text{bMaj } T' - \text{bMaj } T = j + q.$$

It suffices now to remark that $\ell = j + q$.

- (4) $t + 1 \leq \ell \leq r$. Note that a_ℓ is a position before a complete block. Let q and s be the greatest numbers such that $d_q > a_\ell$ and $o_s > a_\ell$. Clearly, $\text{rsb}_i(T') = s$ and the block descent set of T' is

$$\{d_p < \dots < d_{q+1} < a_\ell + 1 < d_q + 1 < d_{q-1} + 1 < \dots < d_1 + 1\}.$$

It follows that

$$\text{rsb}_i T' + \text{bMaj } T' - \text{bMaj } T = s + q + a_\ell + 1.$$

It suffices now to remark that $a_\ell = \ell - (s + q) - 1$. Indeed, a_ℓ is equal to the number of positions in T to the left of the position, i.e. $(p - q) + (m - s) + (\ell - t - 1)$, which is also equal to $\ell - (q + s) - 1$ since $t = p + m$.

□

We give an example to illustrate the above result.

Example 8.1. Let $T = 6 \ 11 \ \infty / 3 \ 5 \ 7 / 1 \ 4 \ 10 \ \infty / 9 / 2 \ 8$. So there are 6 insertion positions and the set of active positions and descents is $A = \{0, 2\}$ and $D = \{4\}$. Therefore

$$a_0 = 5, \ a_1 = 4, \ a_2 = 2, \ a_3 = 0, \quad \text{and} \quad a_4 = 1, \ a_5 = 3.$$

Denote by T' the trace obtained by inserting in T the block $\{12\}$ into position a_i . Then we have:

i	a_i	T'	$\text{rsb}_{12}(T') + \text{bMaj}(T') - \text{bMaj}(T)$
0	5	6 11 $\infty/3$ 5 7/1 4 10 $\infty/9/2$ 8/ 12	$0 + 4 - 4 = 0$
1	4	6 11 $\infty/3$ 5 7/1 4 10 $\infty/9$ / 12 /2 8	$0 + 5 - 4 = 1$
2	2	6 11 $\infty/3$ 5 7/ 12 /1 4 10 $\infty/9/2$ 8	$1 + 5 - 4 = 2$
3	0	12 /6 11 $\infty/3$ 5 7/1 4 10 $\infty/9/2$ 8	$2 + 5 - 4 = 3$
4	1	6 11 ∞ / 12 /3 5 7/1 4 10 $\infty/9/2$ 8	$1 + 7 - 4 = 4$
5	3	6 11 $\infty/3$ 5 7/1 4 10 ∞ / 12 /9/2 8	$0 + 9 - 4 = 5$

We now construct a new bijection $\Psi : \Delta_n^k \mapsto \mathcal{OP}_n^k$ based on the above lemma.

Algorithm Ψ . Let $h = (w, \gamma) \in \Delta_n^k$ be a path diagram. Set $T_0 = \emptyset$. Construct recursively i -skeletons T_i for $i = 1, \dots, n$ such that T_i has $y_{i+1}(w)$ active blocks and $x_{i+1}(w)$ complete blocks by the following process. Suppose $T_{i-1} = B_1/B_2/\dots/B_\ell$ and T_{i-1} has $y_i(w)$ active blocks and $x_i(w)$ complete blocks. Label the positions before B_1 , between B_j and B_{j+1} , for $1 \leq j \leq \ell - 1$, and after B_ℓ from left to right by $\{0, 1, \dots, \ell\}$. Extend T_{i-1} to T_i as follows:

- *The i -th step of w is North (resp. East):* Let A be the set of the active positions in T_{i-1} and D the set of block descents in T_{i-1} . Then, set $a_0 = \ell$, $\{a_1 > a_2 > \dots > a_t\} = A \cup D$ and let $a_{t+1} < \dots < a_\ell$ be the remaining positions. Then insert the block $\{i, \infty\}$ (resp. $\{i\}$) into position a_{γ_i} .
- *The i -th step of w is Null (resp. South-East):* label the $y_i(w)$ active blocks of T_{i-1} from right to left by $\{0, 1, \dots, y_i(w) - 1\}$. Then insert i (resp. replace ∞ by i) in the active block labeled by γ_i .

Since $x_{n+1}(w) = k$ and $y_{n+1}(w) = 0$, it then follows that T_n has k complete blocks and 0 active blocks, i.e. $T_n \in \mathcal{OP}_n^k$. Define $\Psi(h) = T_n$.

Example 8.2. Let h be the path diagram in Figure 2, then

$$\Psi(h) = 6/3 \ 5 \ 7/9/1 \ 4 \ 10/2 \ 8.$$

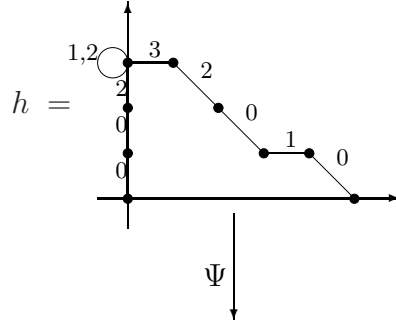
The step by step construction of $\Psi(h)$ is given in Figure 8.

To show that Ψ is bijective, we give its inverse.

Algorithm Ψ^{-1} . Let $\pi \in \mathcal{OP}_n^k$ be an ordered partition. Let w be the path defined by $\lambda(w) = \lambda(\pi)$. Suppose $\emptyset \cup \mathcal{S}(w) = \{i_1 < i_2 < \dots < i_k\}$. Let

$$\gamma_{i_j} = \text{rsb}_{i_j} \pi + \text{bMaj } T_{i_j}(\pi) - \text{bMaj } T_{i_{j-1}}(\pi) \quad \text{for } i = 1 \dots k$$

and $\gamma_i = \text{rsb}_i(\pi)$ if $i \in \mathcal{T} \cup \mathcal{S}(w)$. It is not hard to verify that $(w, (\gamma_i)_{1 \leq i \leq n})$ is a walk diagram in Ω_n^k and its image under Ψ is π .



i	step_i	γ_i	T_i
0			\emptyset
1	<i>North</i>	0	$\mathbf{1} \infty$
2	<i>North</i>	0	$\mathbf{1} \infty / \mathbf{2} \infty$
3	<i>North</i>	2	$\mathbf{3} \infty / \mathbf{1} \infty / \mathbf{2} \infty$
4	<i>Null</i>	1	$\mathbf{3} \infty / \mathbf{1} \mathbf{4} \infty / \mathbf{2} \infty$
5	<i>Null</i>	2	$\mathbf{3} \mathbf{5} \infty / \mathbf{1} \mathbf{4} \infty / \mathbf{2} \infty$
6	<i>East</i>	3	$\mathbf{6} / \mathbf{3} \mathbf{5} \infty / \mathbf{1} \mathbf{4} \infty / \mathbf{2} \infty$
7	<i>South-East</i>	2	$\mathbf{6} / \mathbf{3} \mathbf{5} \mathbf{7} / \mathbf{1} \mathbf{4} \infty / \mathbf{2} \infty$
8	<i>South-East</i>	0	$\mathbf{6} / \mathbf{3} \mathbf{5} \mathbf{7} / \mathbf{1} \mathbf{4} \infty / \mathbf{2} \mathbf{8}$
9	<i>East</i>	1	$\mathbf{6} / \mathbf{3} \mathbf{5} \mathbf{7} / \mathbf{9} / \mathbf{1} \mathbf{4} \infty / \mathbf{2} \mathbf{8}$
10	<i>South-East</i>	0	$\mathbf{6} / \mathbf{3} \mathbf{5} \mathbf{7} / \mathbf{9} / \mathbf{1} \mathbf{4} \mathbf{10} / \mathbf{2} \mathbf{8}$

FIGURE 8. the step by step construction of $\Psi(h)$

By definition of Ψ , for $1 \leq i \leq n$, $\text{bMaj } T_i(\pi) - \text{bMaj } T_{i-1}(\pi) = 0$ for any $i \in \mathcal{T} \cup \mathcal{C}(\pi)$. Since $\text{bMaj } \emptyset = 0$ and $T_n(\pi) = \pi$, splitting the set $[n]$ into $\emptyset \cup \mathcal{S}(\pi)$ and $\mathcal{T} \cup \mathcal{C}(\pi)$ we see that

$$\text{bMaj } \pi = \sum_{i \in \emptyset \cup \mathcal{S}(\pi)} (\text{bMaj } T_i(\pi) - \text{bMaj } T_{i-1}(\pi)). \quad (8.2)$$

We summarize the main properties of Ψ in the following theorem.

Theorem 8.2. *The mapping $\Psi : \Delta_n^k \mapsto \mathcal{OP}_n^k$ is a bijection such that if $h = (w, \gamma) \in \Delta_n^k$ and $\pi = \Psi(h)$, then $\lambda(\pi) = \lambda(w)$ and*

$$\gamma_i = \begin{cases} \text{rsb}_i(\pi) + \text{bMaj}(T_i(\pi)) - \text{bMaj}(T_{i-1}(\pi)), & \text{if } i \in \emptyset \cup \mathcal{S}(\pi); \\ \text{rsb}_i(\pi), & \text{if } i \in \mathcal{T} \cup \mathcal{C}(\pi). \end{cases} \quad (8.3)$$

Therefore

$$\sum_{i \in \mathcal{O} \cup \mathcal{S}(w)} \gamma_i = \text{rsb}_{\mathcal{OS}} \pi + \text{bMaj} \pi, \quad \text{and} \quad \sum_{i \in \mathcal{T} \cup \mathcal{C}(w)} \gamma_i = \text{rsb}_{\mathcal{TC}} \pi. \quad (8.4)$$

9. PROOF OF THEOREM 3.3

Consider the mapping

$$\Upsilon := \Psi \circ \Phi^{-1} : \mathcal{OP}_n^k \rightarrow \mathcal{OP}_n^k.$$

For example, if $\pi = 6/3 \ 5 \ 7/1 \ 4 \ 10/9/2 \ 8$, then it follows from Figure 2 and Figure 3 that $\Upsilon(\pi) = 6/3 \ 5 \ 7/9/1 \ 4 \ 10/2 \ 8$. Note that $\Upsilon(\pi)$ is generally not a rearrangement of the blocks of π .

The main properties of Υ is summarized in the following lemma.

Lemma 9.1. *The map $\Upsilon : \mathcal{OP}_n^k \mapsto \mathcal{OP}_n^k$ is a bijection such that for any $\pi \in \mathcal{OP}_n^k$, we have:*

- (i) $\lambda(\Upsilon(\pi)) = \lambda(\pi)$,
- (ii) $\text{rsb}_{\mathcal{TC}} \Upsilon(\pi) = \text{rsb}_{\mathcal{TC}} \pi$,
- (iii) $\text{MAJ} \Upsilon(\pi) = \text{INV} \pi$.

Proof. By definition the mapping Υ is a bijection. For $\pi \in \mathcal{OP}_n^k$, let $h = (w, \gamma) = \Phi^{-1}(\pi)$ and $\pi' = \Psi(h)$. Hence $\pi' = \Upsilon(\pi)$. By the construction of Ψ and Φ , it is clear that $\lambda(\pi') = \lambda(w) = \lambda(\pi)$. Combining (4.1), (8.4) and (3.7), we have

$$\begin{aligned} \text{MAJ} \pi' &= \text{rsb}_{\mathcal{OS}} \pi + \text{bMaj} \pi' = \sum_{i \in \mathcal{O} \cup \mathcal{S}(w)} \gamma_i = \text{ros}_{\mathcal{OS}} \pi = \text{INV} \pi, \\ \text{rsb}_{\mathcal{TC}} \pi' &= \sum_{i \in \mathcal{T} \cup \mathcal{C}(w)} \gamma_i = \text{rsb}_{\mathcal{TC}} \pi, \end{aligned}$$

completing the proof. □

Applying Lemma 9.1 and Proposition 6.4 we obtain

$$\begin{aligned} &(\text{cls} + \text{rsb}_{\mathcal{TC}}, \text{opb} + \text{rsb}_{\mathcal{TC}}, \text{sb} - \text{rsb}_{\mathcal{TC}}, \text{MAJ}) \Upsilon(\pi) \\ &= (\text{cls} + \text{rsb}_{\mathcal{TC}}, \text{opb} + \text{rsb}_{\mathcal{TC}}, \text{sb} - \text{rsb}_{\mathcal{TC}}, \text{INV}) \pi. \end{aligned} \quad (9.1)$$

According to (2.5) and (2.7), the following functional identities hold on \mathcal{OP}_n^k :

$$\begin{aligned} \text{MAK} + \text{bMaj} &= \text{lcs} + \text{rcs} + \text{rsb} + \text{bMaj} \\ &= \text{cls} + \text{rsb}_{\mathcal{TC}} + \text{rsb}_{\mathcal{OS}} + \text{bMaj} \\ &= \text{cls} + \text{rsb}_{\mathcal{TC}} + \text{MAJ}, \\ \text{MAK}' + \text{bMaj} &= \text{lob} + \text{rob} + \text{rsb} + \text{bMaj} \\ &= \text{opb} + \text{rsb}_{\mathcal{TC}} + \text{rsb}_{\mathcal{OS}} + \text{bMaj} \\ &= \text{opb} + \text{rsb}_{\mathcal{TC}} + \text{MAJ}, \\ \text{cmajLSB} &= k(k-1) + \text{sb} - \text{rsb} - \text{bMaj} \\ &= k(k-1) + \text{sb} - \text{rsb}_{\mathcal{TC}} - \text{rsb}_{\mathcal{OS}} - \text{bMaj} \\ &= k(k-1) + \text{sb} - \text{rsb}_{\mathcal{TC}} - \text{MAJ}. \end{aligned}$$

We thus derive from (9.1) and (7.4) that

$$\begin{aligned} & (\text{MAK} + \text{bMaj}, \text{MAK}' + \text{bMaj}, \text{cmajLSB})\pi' \\ &= (\text{MAK} + \text{bInv}, \text{MAK}' + \text{bInv}, \text{cinvLSB})\pi. \end{aligned}$$

This completes the proof of Theorem 3.3.

Remark 9.2. From (7.8) and (9.1) we derive immediately the following equivalent forms of (3.4):

$$\begin{aligned} \sum_{\pi \in \mathcal{OP}_n^k} p^{\text{cls } \pi + \text{rsb}_{\mathcal{TC}} \pi} q^{\text{sb } \pi - \text{rsb}_{\mathcal{TC}} \pi} t^{\text{MAJ} \pi} &= [k]_t! S_{p,q}(n, k), \\ \sum_{\pi \in \mathcal{OP}_n^k} p^{\text{opb } \pi + \text{rsb}_{\mathcal{TC}} \pi} q^{\text{sb } \pi - \text{rsb}_{\mathcal{TC}} \pi} t^{\text{MAJ} \pi} &= [k]_t! S_{p,q}(n, k). \end{aligned} \tag{9.2}$$

Remark 9.3. Composing Υ and Ξ we obtain the mapping

$$\Theta = \Psi \circ \varphi \circ \Psi^{-1} : \mathcal{OP}_n^k \longrightarrow \mathcal{OP}_n^k.$$

For example, if $\pi = 6/3 \ 5 \ 7/9/1 \ 4 \ 10/2 \ 8$, then $\Psi^{-1}(\pi) = h$, where h is the path diagram in Figure 8. Therefore $\varphi(h)$ is that given in Example 5.1. The reader can verify that $\Theta(\pi) = \Psi(\varphi(h)) = 4 \ 6 \ 8/1 \ 7 \ 10/3 \ 9/5/2$.

Obviously, the mapping Θ is an involution on \mathcal{OP}_n^k satisfying, for any $\pi \in \mathcal{OP}_n^k$,

- (i) $\lambda(\Theta(\pi)) = \overline{\lambda(\pi)}$,
- (ii) $\text{rsb}_{\mathcal{TC}} \Theta(\pi) = \text{rsb}_{\mathcal{TC}} \pi$,
- (iii) $\text{MAJ} \Theta(\pi) = \text{MAJ} \pi$.

Therefore, by Proposition 6.4,

$$(\text{opb} + \text{rsb}_{\mathcal{TC}}, \text{sb} - \text{rsb}_{\mathcal{TC}}, \text{MAJ}) \Theta(\pi) = (\text{cls} + \text{rsb}_{\mathcal{TC}}, \text{sb} - \text{rsb}_{\mathcal{TC}}, \text{MAJ}) \pi.$$

In other words, we have

$$\begin{aligned} & (\text{MAK} + \text{bMaj}, \text{MAK}' + \text{bMaj}, \text{cmajLSB}) \Theta(\pi) \\ &= (\text{MAK}' + \text{bMaj}, \text{MAK} + \text{bMaj}, \text{cmajLSB}) \pi. \end{aligned} \tag{9.3}$$

10. PROOF OF THEOREM 3.5

For any $\pi = B_1/B_2/\dots/B_k$ in \mathcal{P}_n^k , we have

$$\mathcal{R}(\pi) = \{B_{\sigma(1)}/B_{\sigma(2)}/\dots/B_{\sigma(k)} \mid \sigma \in S_k\}.$$

By (2.6) and (3.7), we have $\text{INV}(B_{\sigma(1)}/B_{\sigma(2)}/\dots/B_{\sigma(k)}) = \text{inv } \sigma$. It follows by using Lehmer code (see Section 6) that for any $\pi \in \mathcal{P}_n^k$,

$$\sum_{\pi \in \mathcal{R}(\pi)} q^{\text{INV} \pi} = \sum_{\sigma \in S_k} q^{\text{inv } \sigma} = [k]_q!. \tag{10.1}$$

It remains to show that

$$\sum_{\pi \in \mathcal{R}(\pi)} q^{\text{MAJ} \pi} = [k]_q!. \tag{10.2}$$

Let $C_k = \{(c_1, \dots, c_k) : 0 \leq c_i \leq i-1\}$. We will construct a bijection $\beta_\pi : C_k \mapsto \mathcal{R}(\pi)$ such that for any $\mathbf{c} = (c_1, \dots, c_k) \in C_k$, we have $\text{MAJ} \beta_\pi(\mathbf{c}) = \sum_{i=1}^k c_i$.

We construct recursively i -skeletons T_i , $1 \leq i \leq n$, such that T_i has $\mathcal{O}(\pi)^{\leq i} - \mathcal{C}(\pi)^{\leq i}$ active blocks and $(\mathcal{C} \cup \mathcal{S}(\pi))^{\leq i}$ complete blocks by the following process. Set $T_0 = \emptyset$ and suppose $T_{i-1} = B_1/B_2/\cdots/B_l$. Then T_i is obtained from T_{i-1} as follows:

- $i \in \mathcal{O}(\pi)$ (resp. $\mathcal{S}(\pi)$): label the positions before B_1 , between B_j and B_{j+1} , for $1 \leq j \leq l-1$, and after B_l from left to right by $\{0, 1, \dots, l\}$.

Let F be the set of the positions before the active blocks in T_{i-1} and $D = \text{bDes}(T_{i-1})$. Then, set $a_0 = l$, $\{a_1 > a_2 > \cdots > a_t\} = F \cup D$ and let $a_{t+1} < \cdots < a_l$ be the remaining positions. We then insert the block $\{i, \infty\}$ (resp. $\{i\}$) into position a_{γ_i} .

- $i \in \mathcal{T}(\pi)$ (resp. $\mathcal{C}(\pi)$): insert i (resp. replace ∞ by i) in the active block whose opener is the opener of the block of π which contains i .

It is not difficult to see, via Lemma 8.1, that the above procedure is well defined. Since $\mathcal{O}(P)^{\leq n} = \mathcal{C}(P)^{\leq n}$ and $(\mathcal{C}(P) \cup \mathcal{S}(P))^{\leq n} = |\mathcal{C}(P) \cup \mathcal{S}(P)| = k$, T_n is a n -skeleton with 0 active blocks and k complete blocks, i.e. $T_n \in \mathcal{OP}_n^k$. Now, by construction, $T_n \in \mathcal{R}(\pi)$. We then set $\beta_P(\mathbf{c}) = T_n$.

To show that β_P is bijective, we describe its inverse. Let $\pi \in \mathcal{R}(\pi)$ and suppose

$$\mathcal{O} \cup \mathcal{S}(\pi) = \{i_1 < i_2 < \cdots < i_k\}.$$

For $1 \leq j \leq k$, let $c_j = \text{rsb}_{i_j} \pi + \text{bMaj } T_{i_j}(\pi) - \text{bMaj } T_{i_{j-1}}(\pi)$. It is then readily seen that $\beta_P(c_1 \cdots c_k) = \pi$.

Remark. A Foata style bijection which establishes directly the equidistribution of INV and MAJ will be given by the first author in [12].

11. CONCLUDING REMARKS

Recall that the q -Eulerian numbers $A_q(n, k)$ ($n \geq k \geq 0$) of Carlitz [4] are defined by

$$A_q(n, k) = q^k [n - k]_q A_q(n - 1, k - 1) + [k + 1]_q A_q(n - 1, k), \quad (11.1)$$

and have the following combinatorial interpretation:

$$A_q(n, k) = \sum_{\sigma} q^{\text{maj } \sigma}$$

where the summation is over all permutations of $[n]$ with k descents.

The original motivation of Steingrímsson [19] was to give a direct combinatorial proof of the following identity [24]:

$$[k]_q! S_q(n, k) = \sum_{m=1}^k q^{k(k-m)} \begin{bmatrix} n - m \\ n - k \end{bmatrix}_q A_q(n, m - 1). \quad (11.2)$$

Though we have proved all the conjectures inspired by (11.2), a direct combinatorial proof of (11.2) is still *missing*. As proved in [24] the identity (11.2) is equivalent to Garsia's q -analogue of Frobenius formula relating q -Eulerian numbers and q -Stirling numbers of the second kind (see [8, 9]):

$$\sum_{k=1}^n \frac{[k]_q! S_q(n, k) x^k}{(x; q)_{k+1}} = \sum_{k=1}^{\infty} [k]_q^n x^k = \frac{\sum_{\sigma \in S_n} x^{1+\text{des } \sigma} q^{\text{maj } \sigma}}{(x; q)_{n+1}}. \quad (11.3)$$

Note that a combinatorial proof of (11.3) has been given by Garsia and Remmel [9].

Briggs and Remmel [2] proved the following p, q -analogue of Frobenius formula (11.3):

$$\sum_{k=1}^n \frac{[k]_{p,q}! \hat{S}_{p,q}(n, k) p^{\binom{n-k+1}{2} + k(n-k)} x^k}{(xp^n; q/p)_{k+1}} = \frac{\sum_{\sigma \in S_n} x^{\text{des } \sigma + 1} q^{\text{maj } \sigma} p^{\text{comaj } \sigma}}{(xp^n; q/p)_{n+1}}, \quad (11.4)$$

where $\text{comaj } \sigma = n \text{ des } \sigma - \text{maj } \sigma$ and $\hat{S}_{p,q}(n, k)$ is a variante of the p, q -Stirling numbers of the second kind defined by the following recursion:

$$\hat{S}_{p,q}(n, k) = q^{k-1} \hat{S}_{p,q}(n-1, k-1) + p^{-n} [k]_{p,q} \hat{S}_{p,q}(n-1, k). \quad (11.5)$$

We would like to point out that (11.4) and (11.3) are equivalent. Obviously (11.3) corresponds to the $p = 1$ case of (11.4). Conversely, since $[k]_{q/p}! = p^{-\binom{k}{2}} [k]_{p,q}!$ and

$$\begin{aligned} \hat{S}_{p,q}(n, k) &= p^{-\binom{n-k+1}{2} - (n-k)} q^{\binom{k}{2}} S_{1,q/p}(n, k) \\ &= p^{\binom{k}{2} - \binom{n-k+1}{2} - (n-k)} S_{q/p}(n, k), \end{aligned}$$

we derive (11.4) from (11.3) by substituting $q \rightarrow q/p$ and $x \rightarrow xp^n$.

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